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Bruno Solnik

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BRUNO SOLNIK*

INTERNATIONAL ASSET PRICING (IAPM) has been the object of an intense controversy due to different assumptions on utility functions, sources of price uncertainty, and market imperfections. Some models assumed that all investors consume the same good, with different stochastic national inflation rates,[1] while others model a world in which exchange rates reflect relative price changes (deviation from purchasing power parity) with non stochastic inflation and different consumption tastes across countries.[2] Stülz (14) proposed a fairly complete consumption based model with nominal riskless bonds in every country, while a survey by Adler and Dumas (1) offers the most comprehensive and clarifying analysis of the IAPM.

A general conclusion is that the world market portfolio will not be optimal in the sense that investors will hold different portfolios, especially "hedge" portfolios (Solnik (10), Stülz (14), Adler and Dumas (1)). Since the composition of these portfolios depends on the covariance of asset returns with state variables, it is hard to identify such portfolios in order to test the theory. The attractive and simple domestic CAPM conclusion that a well identified market portfolio is efficient does not exist in the international framework, so that the IAPM does not yield operational (and easily testable) conclusions.

The Arbitrage Pricing Theory formulated by Ross (8, 9) provides a fruitful alternative to these utility based models. International Arbitrage Pricing Theory (IAPT) only requires perfect capital markets. It is shown below that the numeraires used by investors to measure (real) returns do not have to be specified[3] so long as they believe (homogeneously) that nominal asset returns follow a m-factor generating model. In other words, the differences between national investors need not be modeled.

As stressed by Ross, the m-factor assumption replaces the multivariate normal or Ito-Wiener asset return distribution assumption of the IAPM. In asset pricing models, "mathematical" difficulties arise when asset demands are aggregated over people using different numeraires to measure returns; such a problem is not present with APT because common factors are not constrained to be weighted "averages" of individual assets as are portfolios.

While IAPM uses the international parlance, it should be stressed that the analysis differs from the traditional CAPM by the introduction of differences in

* Centre D'Enseignement Superieur des Affaires, 78350 Jouy en Josas, France.

[1] See for instance Grauer, Litzenberger and Stehle (4), Kouri (5) and Fama and Farber (3).

[2] See for instance Solnik (10) and (11).

[3] All that is required is that investors be able to define some real deflator to apply to nominal returns.

consumption tastes and relative price uncertainty.[4] The IAPM is actually a multi-consumption real CAPM. Similarly, the IAPT results derived here could be regarded as an extension of the nominal APT to accommodate diverse consumption tastes and relative price uncertainty. While the international framework will be used throughout this paper, the reader should be aware that all the results could be stated in terms of real vs nominal rather than international vs domestic.

1—*The Usual APT on Nominal Returns*

Let's assume that there exist $n + 1$ currencies, and $N + 1$ assets with N much larger than n. One currency, say the U.S. Dollar, is arbitrarily chosen as the nominal numeraire and numbered 0. For the time being, assume that the first $n + 1$ assets are national bills, riskless in local currency terms; the asset subscripted 0 is the nominal riskfree asset in currency 0 (dollar). Investors are assumed to believe that the random returns on the set of assets are governed over short intervals of time by a m-factor generating model of the form:[5]

$$\tilde{r}_i = E_i + b_{i1}\tilde{\delta}_1 + \cdots + b_{im}\tilde{\delta}_m + \tilde{\epsilon}_i \qquad i = 1, \cdots, N \tag{1}$$

where E_i is the expected return on the ith asset. The m zero mean common factors δ capture all systematic risks while the noise term ε_i is idiosyncratic to asset i. The ϵ's reflect all information unrelated to other assets and therefore are assumed to be independent of each other as well as unrelated to the common factors δ:

$$E(\tilde{\epsilon}_i \mid \tilde{\delta}_k) = 0 \quad \text{for all } i \text{ and } k.$$

We further assume that the number of common factors is much smaller than the number of assets.

The usual derivation of the nominal APT can now be performed for an investor who cares about U.S. dollar returns. To determine a pricing relation, one can build a well diversified arbitrage portfolio with weights x_i invested in asset i so that:

$$\begin{aligned} \textstyle\sum_i x_i &= \underline{X}\underline{1} = 0 \\ \textstyle\sum_i x_i b_{ik} &= \underline{X}\underline{b_k} = 0 \quad k = 1, \cdots, m \\ \textstyle\sum x_i \tilde{\epsilon}_i &= \underline{X}\underline{\epsilon} \sim 0 \end{aligned} \tag{2}$$

where $\underline{X}$ is a row vector and $\underline{\tilde{r}}$, $\underline{E}$, $\tilde{\epsilon}$, $\underline{b_k}$, $\underline{1}$ are column vectors of size N.

This arbitrage portfolio is constructed as well diversified in the sense that the residual risk is negligible. The return on this arbitrage portfolio is equal to:

$$\underline{X\tilde{r}} = \underline{XE} + \textstyle\sum_k \underline{Xb_k}\tilde{\delta}_k + \underline{X\tilde{\epsilon}} \sim \underline{XE} \tag{3}$$

[4] At least, the more comprehensive theories allowing for different consumption goods.

[5] For a more detailed description of the now well-known assumptions and implications of APT, see Ross (8, 9) and Roll and Ross (7).

This portfolio is almost riskless and, since it has a zero capital investment, it should also have a zero return (so that $\underline{XE} = 0$). As pointed out by Ross, this implies that the vector of expected return E must be a linear combination of the constant vector $\underline{1}$ and the $\underline{b_k}$ vectors, i.e., there exists $m + 1$ scalar constants $\lambda_0 \cdots \lambda_m$, such that:

$$\underline{E} = \lambda_0 + \lambda_1 \underline{b_1} + \cdots + \lambda_m \underline{b_m} \tag{4}$$

Since there exists a U.S. riskfree asset, its return r_0 is equal to λ_0.

We will now derive the equivalent of relation (4) for a foreign investor and aggregate it to a testable market relationship.

2—*International APT*

Investors of different countries measure (or care about) returns in different units, i.e. currencies; they are assumed to adjust nominal (dollar) returns by a random variable $\tilde{s}$. To use the international terminology, we will call this an exchange rate adjustment but this formulation also applies to the case where any investor j observes nominal returns and adjusts them by a specific inflation deflator $\tilde{s}_j$.

At first, one would not expect differences in utility functions to affect Arbitrage Pricing results as long as investors hold homogeneous expectations on the return generating model in nominal terms.[6] However, while APT is not a utility based approach, it does require the definition of a riskless investment. For example, if all investors are subject to stochastic inflation, they will not regard a nominal riskless portfolio as riskfree (i.e. in real terms) and the arbitrage argument applied in Section 1 to derive the pricing relation (4) might not be valid. So we will *first* show that, if asset returns are believed to follow (1), then any *arbitrage* portfolio which is nominally riskless will be riskless for any foreign investor. A *second* and more important question lies in the internal consistency of the m-factor model assumption in an international framework. To be a viable theory, it must be independent of the numeraire arbitrarily chosen to identify the model so that the m-factor model holds from any other currency viewpoint. For example, the m-factor model might hold true in terms of Poupou dollars (a small island on Mars) but, if no investors live in Poupou, it does not do us much good except if a similar m-factor model governs returns measured in other currencies. In other words, the dollar returns of Japanese, French and American stocks should exhibit the same structure as the returns of the same stocks computed in other currencies (Japanese Yen, French Franc or British Pound) and not be an artefact due to the return translation in some particular currency. Let's now prove these two points.

If P_i is the dollar price of asset i and S_{j0} the exchange rate of currency j in units of currency 0, the currency j price of asset i is equal to P_i/S_{j0} and the return

[6] For example, if all investors agree on the dollar factor generating model, and one investor cares about nominal dollar returns, the pricing relation (4) will hold for him as well as all other investors since, by assumption, expectations are homogeneous.

of asset i measured in currency j, for very short time interval,[7] will be equal, by Ito's lemma, to:

$$\tilde{r}_i^j = \tilde{r}_i - \tilde{s}_j - \tilde{r}_i\tilde{s}_j + \sigma_j^2 \tag{5}$$

where σ_j^2 is the variance of $\tilde{s}_j$ the random variation of S_j and $\tilde{r}_i\tilde{s}_j = C_{ij}$ is the covariance between $\tilde{r}_i$ and $\tilde{s}_j$.

Combining relation (1) and (5) yields the expression of the return on asset i in currency j as:

$$\tilde{r}_i^j = E_i + \sigma_j^2 - \tilde{s}_j + \textstyle\sum_k b_{ik}(\tilde{\delta}_k - \tilde{\delta}_k\tilde{s}_j) + \tilde{\epsilon}_i - \tilde{\epsilon}_i\tilde{s}_j$$

Let's now compute the currency j return of an arbitrage portfolio which verifies conditions (2):

$$\underline{X\tilde{r}^j} = \underline{XE} + \underline{X1}(\sigma_j^2 - \tilde{s}_j) + \textstyle\sum_k \underline{Xb_k}(\tilde{\delta}_k - \tilde{\delta}_k\tilde{s}_j) + \underline{X\tilde{\epsilon}} - \underline{X.\tilde{\epsilon}\tilde{s}}_j$$

From (2), this reduces to:

$$\underline{X\tilde{r}^j} = \underline{XE} - \underline{X.\tilde{\epsilon}\tilde{s}}_j \tag{6}$$

The real return on this portfolio differs from its nominal return given in (3), if the last term cannot be diversified away because of systematic correlation between $\tilde{\epsilon}_i$ and currency j fluctuations. However by assumption, all $\tilde{\epsilon}_i$ are uncorrelated to other assets return including asset j which is the currency j riskfree bill.[8] In dollars, this asset return stochastic component is equal to the random exchange rate movement $\tilde{s}_j$, so that each $\tilde{\epsilon}_i$ is independent of $\tilde{s}_j$. Equation (6) therefore reduces to:

$$\underline{X\tilde{r}^j} = \underline{XE}$$

So, even in real or foreign terms, this arbitrage portfolio bears no risk, so that $\underline{XE}$ must be equal to zero in equilibrium and pricing relation (4) must hold for every investor.

Let's now show that the m-factor framework is invariant to the currency used to express the returns.

The riskfree bill in currency j is one of the assets whose dollar return r_j follows relation (1); its stochastic component is equal to $\tilde{s}_j$ when measured in dollars. This implies that $\tilde{s}_j$ also follows a m-generating factor model[9]:

$$\tilde{s}_j = E(\tilde{s}_j) + \textstyle\sum_k b_{jk}\tilde{\delta}_k + \tilde{\epsilon}_j \tag{7}$$

Replacing in (5) $\tilde{r}_i$ and $\tilde{s}_j$ by their m-factor expression gives:

$$\tilde{r}_i^j = E_i - E(\tilde{s}_j) - C_{ij} + \sigma_j^2 + \textstyle\sum_k (b_{ik} - b_{jk})\tilde{\delta}_k + \tilde{\epsilon}_i - \tilde{\epsilon}_j \tag{8}$$

[7] The m-factor generating model is required to hold over the shortest trading interval. The product of two rates of return will be zero (second order) if they are uncorrelated; this will be the case for $E_i\tilde{s}_j$ because E_i is non stochastic, but generally not for $\tilde{\delta}_k\tilde{s}_j$ which will be equal to $\text{cov}(\tilde{\delta}_k, \tilde{s}_j)$. Note that this short time interval assumption is also required by the standard APT as stressed by Roll and Ross (7).

[8] The case where no riskfree bills exist and currency fluctuations do not follow the m-factor model is studied in the appendix.

[9] $\tilde{s}_j$ differs from $\tilde{r}_j$ by a constant term, the currency j interest rate λ_0^j.

Taking the expected value of $\tilde{r}_i^j$ in equation (5) shows that the constant term on the RHS of (8) is equal to $E(\tilde{r}_i^j)$ which will be denoted as E_i^j. Equation (8) might be rewritten as:

$$\tilde{r}_i^j = E_i^j + \sum_k b_{ik}^j \tilde{\delta}_k + \tilde{\mu}_i \tag{9}$$

with $b_{ik}^j = b_{ik} - b_{jk}$ the new factor loading and

$$\tilde{\mu}_i = \tilde{\epsilon}_i - \tilde{\epsilon}_j$$

Note that all the assumptions of the m-factor model are verified. The noise terms are mutually independent, and uncorrelated to the common factors, since this property holds for $\tilde{\epsilon}_i$ and $\tilde{\epsilon}_j$. The riskfree asset of currency now replaces the currency 0 riskfree asset which becomes risky in terms of currency j. In other words, the decomposition in m-factors plus a noise term is invariant to the currency chosen to compute the returns on this makes the IAPT an attractive and operational framework.[10] A pricing relation such as (4) will hold for returns determinated in currency j or any other currency:

$$\underline{E}^j = \lambda_0^j + \lambda_1^j \underline{b}_1^j + \cdots + \lambda_m^j \underline{b}_m^j \tag{4$'$}$$

where λ_0^j is the currency j riskfree rate.

Without giving tedious mathematical derivations, it should be clear that while the *form* of the relation is invariant, the coefficients $\underline{b}_k^j$ and λ_k^j vary with j. Taking the difference between the two pricing equations, it can be shown that they are linked by the relation:

$$C_{ij} = (\lambda_i - \lambda_1^j) b_{i1} + \cdots + (\lambda_m - \lambda_m^j) b_{im}$$

or:

$$\underline{C}_j = (\lambda_1 - \lambda_1^j) \underline{b}_1 + \cdots + (\lambda_m - \lambda_m^j) \underline{b}_m \tag{10}$$

in particular:

$$\sigma_j^2 = (\lambda_1 - \lambda_1^j) b_{j1} + \cdots + (\lambda_m - \lambda_m^j) b_{jm}$$

In a sense, relation (10) is a pricing relationship à la (4), where the coefficients $(\lambda - \lambda^j)$ indicate the price implication of the covariance structure between asset returns and investor J numeraire. If $\underline{C}_j \equiv 0$, then the risk premia will be identical in both currencies. Note that the assumption of only m common factors and independent residuals imply severe constraints on the currency-asset covariance matrix.

The differences in interest rates on two currencies can easily be derived from the pricing relations. In (4), the expected return on the currency j riskfree asset is:

$$\lambda_0^j + E(\tilde{s}_j) = \lambda_0 + \lambda_1 b_{j1} + \cdots + \lambda_m b_{jm} \tag{11}$$

This implies that the interest rate differential (or forward premium), $\lambda_0 - \lambda_0^j$,

[10] Again this attractive result comes partly from the fact that, contrary to portfolio returns, factors do not have to be translated into foreign currencies when investors measure returns differently. To understand this think of factors such as world real growth, etc...

is equal to the expected currency fluctuation, $E(\tilde{s}_j)$, plus a risk premium. This risk premium depends on the covariance of the currency fluctuations with the same common factors. This result is analogous to the traditional findings of the IAPM.

The case where riskless assets do not exist is discussed in the appendix. This is a situation where asset returns follow the m-factor model but currency fluctuations do not. However a simple riskless arbitrage argument can still be developed based on the construction of riskless portfolios in every currency.

The invariance property of the m-factor model will generally not hold in the absence of riskfree assets in each currency, but simple results still obtain. In a sense, the problem might be bypassed by adding individual currencies as common factors (with factor coefficients possibly equal to zero for many or all assets).[11] Since there exists a large number of assets in every country, the total number of assets will still be much larger than that of common factors. Furthermore, the total number of currencies might be reduced to a smaller set of common (currency) factors.

3—*Conclusions*

In this paper, we have provided an analysis of the international extension of arbitrage pricing theory, where investors value returns of the same asset differently; the same analysis could be applied to a domestic framework with heterogeneous consumption tastes. The technical problems posed by currency translation and aggregation in the international CAPM do not arise in APT since factors are not constrained to be portfolios of the original assets. While APT is not utility based, it requires the definition of a riskless portfolio, hence the numeraire used to measure (real) returns matters. This paper shows that, if a factor model is believed to hold when asset returns are expressed in some arbitrarily chosen currency, this factor structure, as well as its major conclusions, is invariant to the currency chosen. The relation between factor coefficients when measured in different currencies has been investigated, as well as the pricing of the forward exchange rates (or national interest rate differentials) which depend on the covariance of the currency fluctuations with the same common factors.

If investors hold homogeneous expectations as measured in some currency, the same m-factor model and pricing relation will apply for everyone and we can aggregate the ex ante specification to a market testable ex post relation. It follows that, whatever the numeraire used, the pricing relation (5) might be subjected to empirical scrutiny. Again, note that the IAPT says nothing about the size of the risk premia λ_k, nor the number or origin of the common factors, but only specifies the *linearity* of the pricing relation.

[11] Similarly, in the domestic real nominal framework, it would be sufficient to introduce the inflation rate as an additional common factor to maintain the same pricing relation on real return. However, if each investor has different consumption tastes with relative price uncertainty, the problem is serious because the number of investors (different inflation rates) is probably larger than that of assets. We are back to the situation of heterogeneous (real) anticipations discussed in Ross (8). Note, however, that as long as investors believe that *nominal* returns are governed by the m-factors generating model, pricing equation (4) will still hold.

Tests of international asset pricing models have so far been scarce and inconclusive (e.g. Solnik (12), Stehle (13)). Major limitations come from data availability, but also from the technical problems involved in the exact identification of the ex ante efficient portfolios as stressed by Roll (6). Consumption based CAPM are not void of similar types of problems as shown by Cornell (2). When the composition of efficient international portfolios suggested by the IAPM depends on utility functions parameters, the empirical task would even seem hopeless. If international markets segmentation plays a significant role, IAPM hardly provide any useful conclusions. In its more heuristic approach, arbitrage pricing theory seems to offer an attractive alternative.

To be a viable and useful theory, the number of common factors in an IAPT must be small compared to the number of assets. The most simple structure would consist of a few international factors common to all assets. Another extreme would be if the sets of common factors strictly differed across national markets. International factors could also be common within specific types of markets (e.g. all bond markets or all stock markets). A likely situation might be the combination of international factors common to all or specific types of assets plus national factors affecting only domestic assets. Of course if the number of factors is too large, the testability and operationality of IAPT is greatly reduced.[12]

APPENDIX

First, it should be stressed that the strong assumption of APT is that the stochastic return process ($\tilde{r}_i$) of a large number of assets can be reduced to *independent* residuals ($\tilde{\epsilon}_i$) by subtracting a linear combination of a small number of common factors ($\tilde{\delta}_k$). The independence assumption of $\tilde{\epsilon}_i$ and $\tilde{\delta}_k$ is not called for by the theory and is only a simple transformation made for economic interpretation purposes.

Let's now assume that the m-factors generating model is believed to hold in currency 0 but that there do not exist riskless bills in the various currencies. Then pricing equation (4) still holds but not relation (7). Fur convenience, let's write the random process of exchange rate fluctuation $\tilde{s}_j$ as:

$$\tilde{s}_j = E(\tilde{s}_j) + \tilde{\nu}_j \tag{A1}$$

Then, the return of asset i expressed in currency j for continuous time diffusion process described in (5) will be:

$$\begin{aligned} \tilde{r}_i^j &= E_i - E(\tilde{s}_j) - C_{ij} + \sigma_j^2 + \textstyle\sum_k b_{ik}\tilde{\delta}_k + \tilde{\epsilon}_i - \tilde{\nu}_j \\ &= E_i^j + \textstyle\sum_k b_{ik}\tilde{\delta}_k + \tilde{\epsilon}_i - \tilde{\nu}_j \end{aligned} \tag{A2}$$

A straightforward application of the arbitrage portfolio method developed in equation (2) and (3) leads to a pricing relation similar to (4):

$$\underline{E}^j = \lambda_0' + \lambda_1'\underline{b_1} + \cdots + \lambda_m'\underline{b_m} \tag{A3}$$

[12] From a practical viewpoint, note that if the same international factors are common to all assets, the international risk diversification may be achieved by restricting its investments to domestic assets. This would not be the case if factors are segmented along national boundaries.

where $\underline{E}^j$ is the vector of expected returns in units of currency j and the factors coefficients b are exactly the same as before, i.e. computed in currency 0. Intuitively, this "strange" mix comes from the fact that the "currency factor" $\tilde{\nu}_j$ appears with a coefficient equal to 1 in relation (A2) for every asset; this vector of coefficients is identical to the constant vector (one) in the arbitrage portfolio construction and therefore does not increase the vector space. While $\tilde{\nu}_j$ is not orthogonal to $\tilde{\epsilon}_i$, this is not required to establish the pricing equation as was mentioned above. Also note that relation (A3) imposes severe constraint on the covariance structure of asset and currency fluctuations C_{ij}. Given the definition of E_i^j, and substracting (A3) from (4) implies that the vector C_{ij} is a linear combination of the coefficients b_{ik}:

$$\underline{C}_j = \gamma_0 + (\lambda_1 - \lambda_1')\underline{b}_1 + \cdots + (\lambda_m - \lambda_m')\underline{b}_m \tag{A4}$$

with

$$\gamma_0 = \lambda_0 - \lambda_0' + E(\tilde{s}_j) - \sigma_j^2$$

In other words the C_{ij}'s cannot be exogeneous; the number of factors postulated for assets sets the number of degrees of freedom on the covariance structure. If relation (A4) was violated, an investor in country j could build arbitrage portfolios to take advantage of it. Note that, to a constant term, this relation is similar to that found where riskless assets existed (equation (10)). In a sense, investors from country j views the return generating process of all asset returns expressed in their own currency as an $m + 1$ factors model with the additional factor being the random fluctuation in exchange rate $\tilde{\nu}_j$. In equation (A2) this could be formalized by replacing $\tilde{\epsilon}_i$ by the results of its regression on $\tilde{\nu}_j$ conditional on all the other factors.

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